Image credit: NASA/JPL-Caltech

The space elevator

The Physics of Interstellar Travel

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Topics

- Part 1
  - Dynamics: cable equation of (non)-motion
  - Tapered cable
  - Materials
  - Counter-weight
- Part 2
  - Climbers
  - Launching from the elevator
  - Elevator construction

Part 1

Motivation for alternative to rockets

- Rockets are an expensive way of getting payloads into space
  - most of the fuel (and thus energy) is spent lifting the propellant itself
  - the rocket itself is often only used once (e.g. burns up)
  - hydrocarbons are environmentally unfriendly
  - long launch pad turn-around time: can only launch ~10 tonnes payload per week
- Saturn V rocket for Apollo
  - 2950 tonnes total to deliver 45 tonnes to lunar orbit
- SpaceX says it charges \$90M for 8 tonnes to GTO → \$11k per kg
- What's the alternative?
  - don't take your propellant with you

Refresher on Keplerian orbits

radius r, velocity v, period P

Kepler's third law: $r^3 = \frac{GM}{4\pi^2}P^2$

Circular orbits: $v^2 = \frac{GM}{r}$

$$\text{Kinetic energy } K = +\frac{v^2}{2}$$

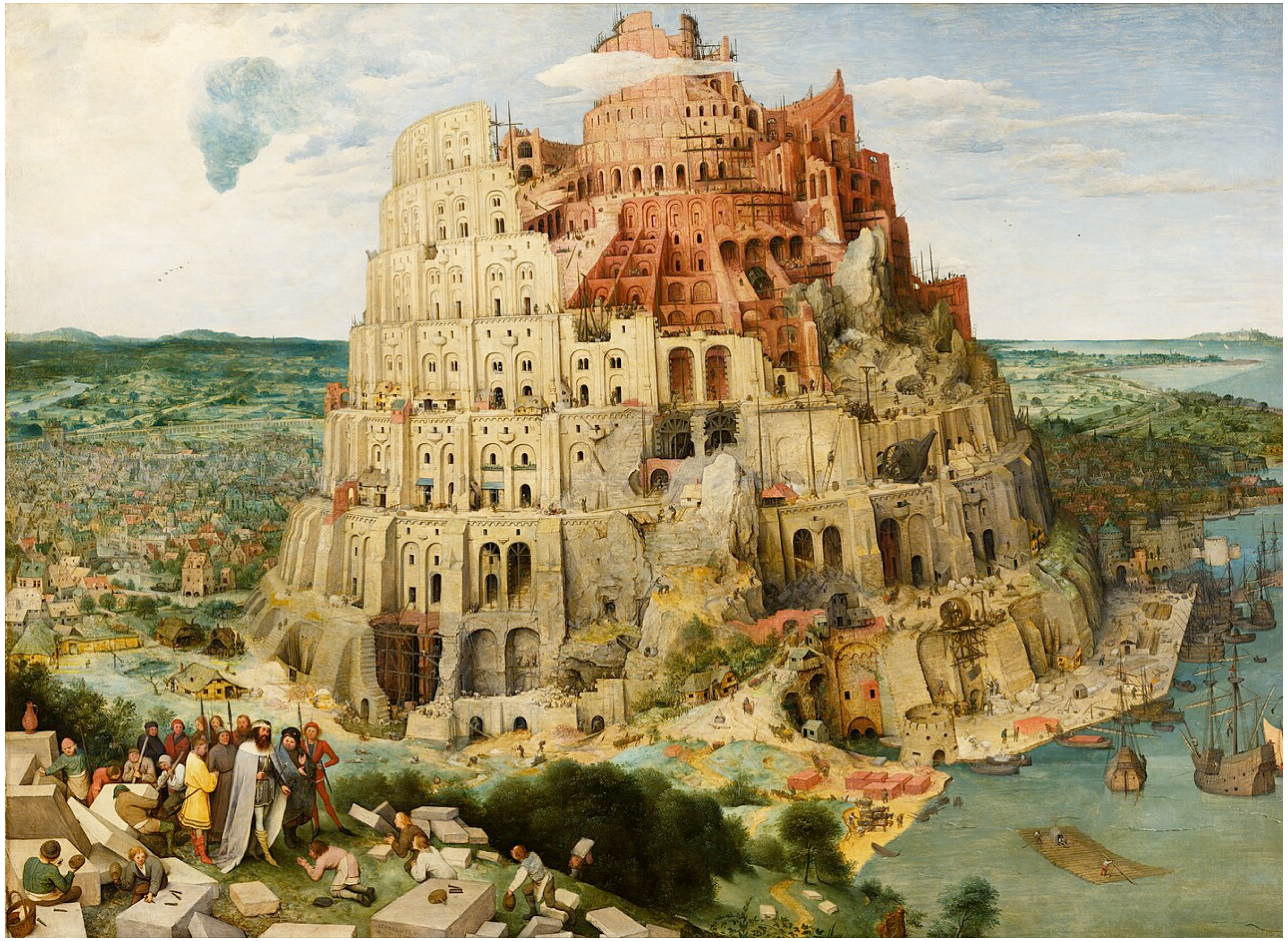
$$\text{Potential energy } U = -\frac{GM}{r}$$

$$\text{Total energy } E = K + U = \frac{U}{2} = -K$$

virial theorem

- Larger orbits have more positive energy
- Objects in larger orbits move more slowly

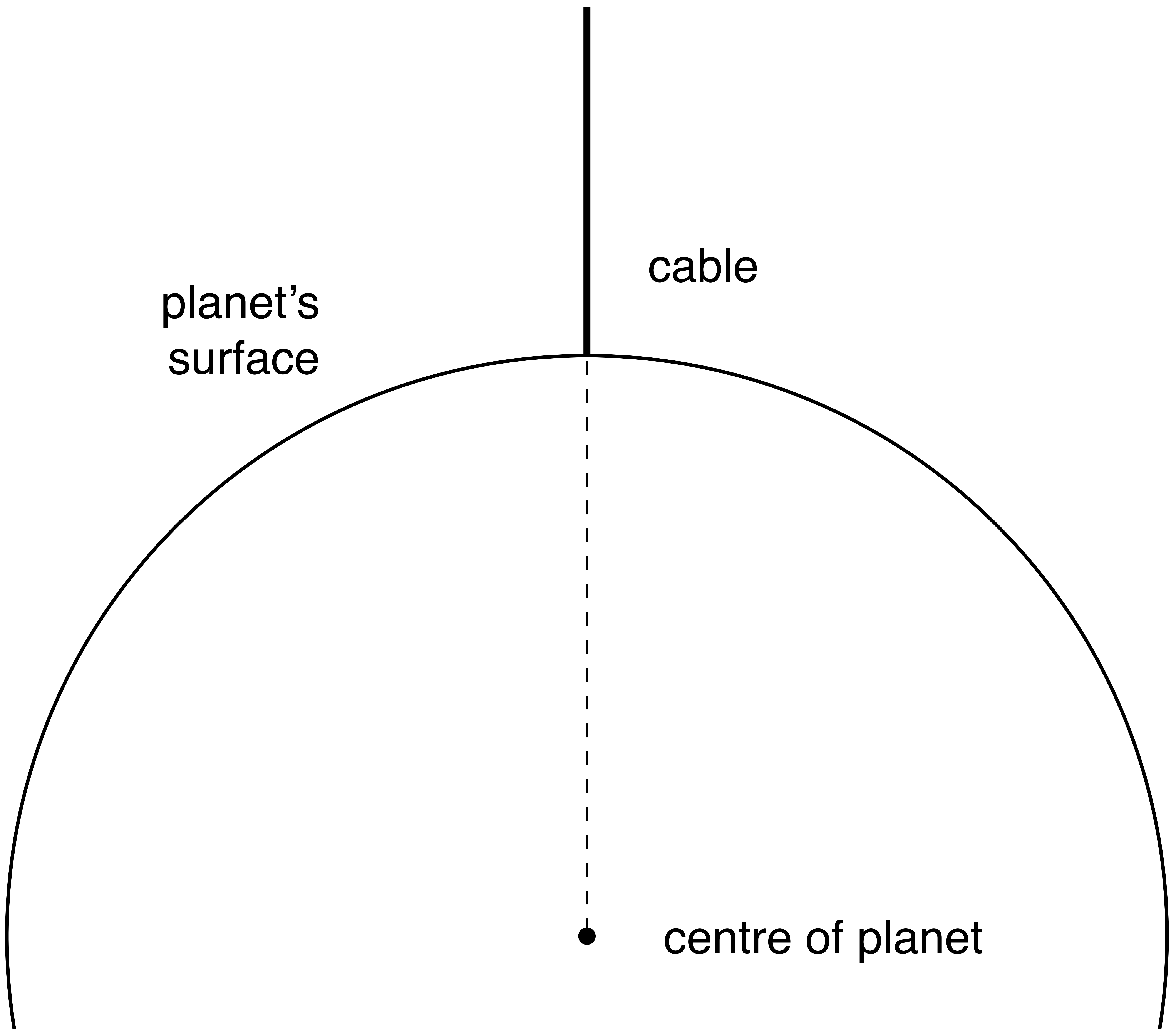
Very tall towers



- Small towers are under compression
- The taller the tower, the more relevant the (apparent) centrifugal force acting upwards

Image credit: Pieter Bruegel the Elder (1563), public domain, from Wikimedia Commons

Space elevator

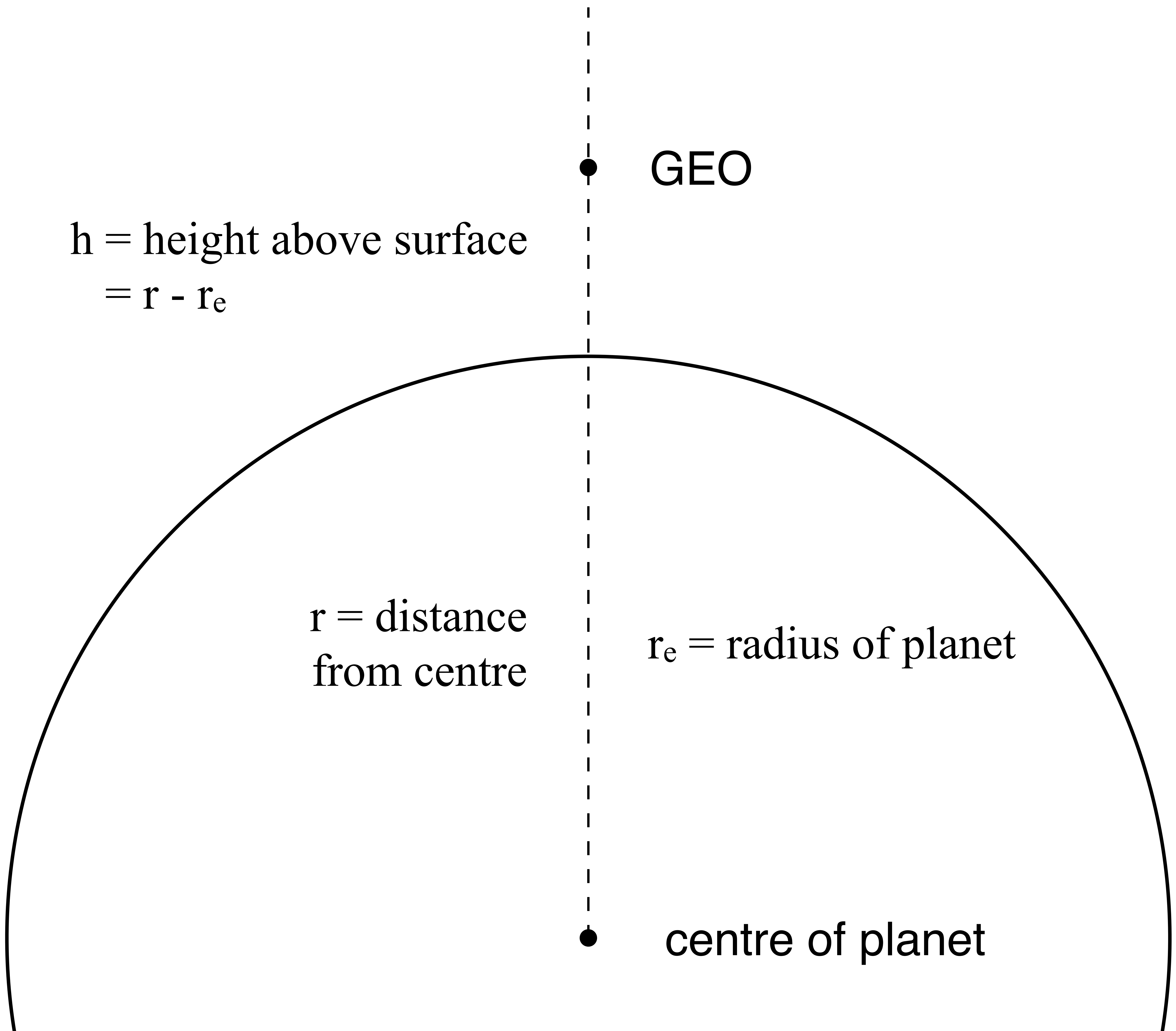


A rigid, straight cable fixed to the planet's surface, extending vertically from the surface.

To be static, the entire cable must rotate at the same rotation period as the planet.

Is it a cable or a tower?

Geostationary orbit (GEO)



Geostationary orbit is circular orbit where the orbital period equals the planet's *sidereal* rotation period, P

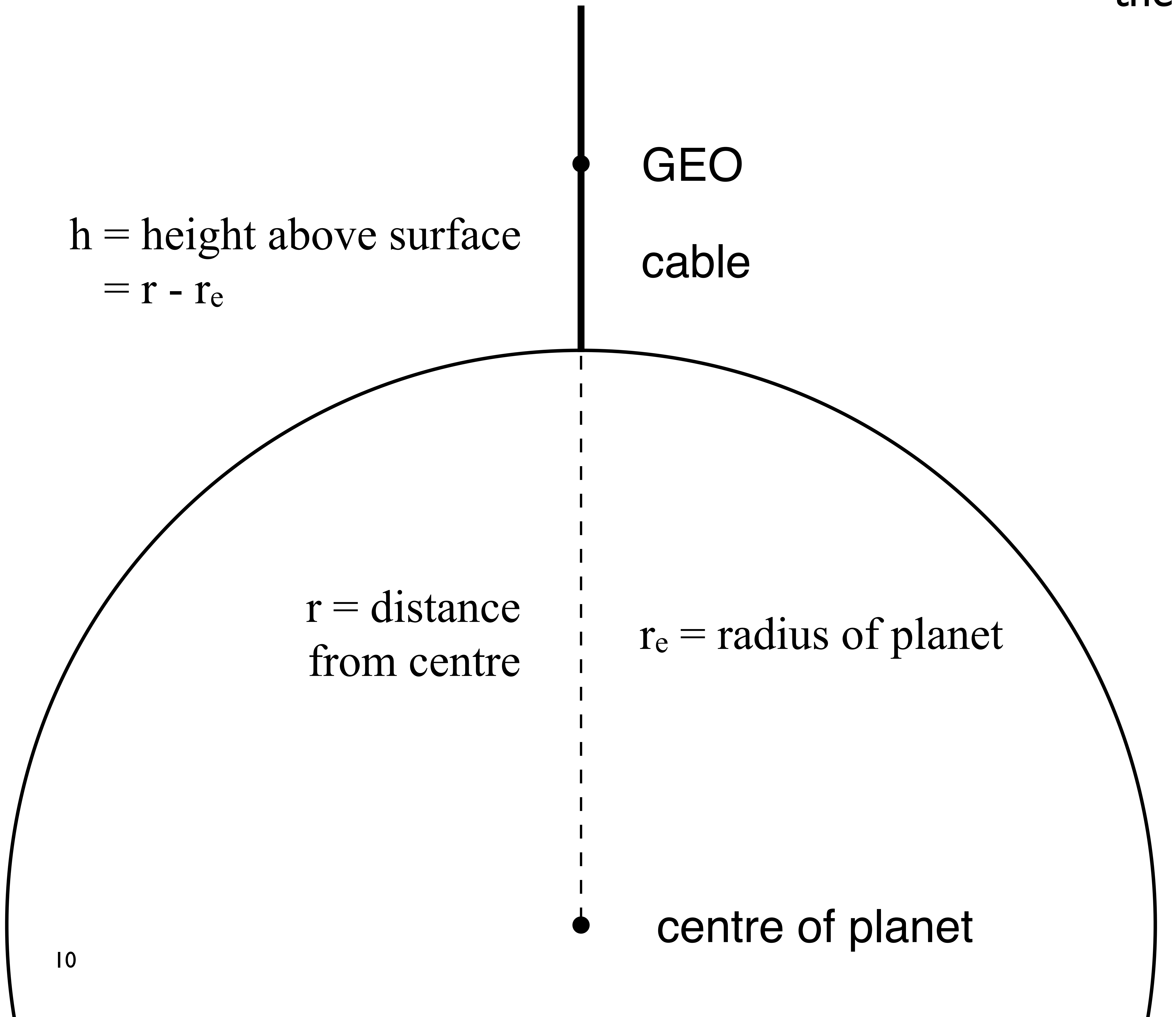
Kepler's third law: $$r_{\text{GEO}}^3 = \frac{GMP^2}{4\pi^2}$$

"Geo"stationary orbits around various bodies

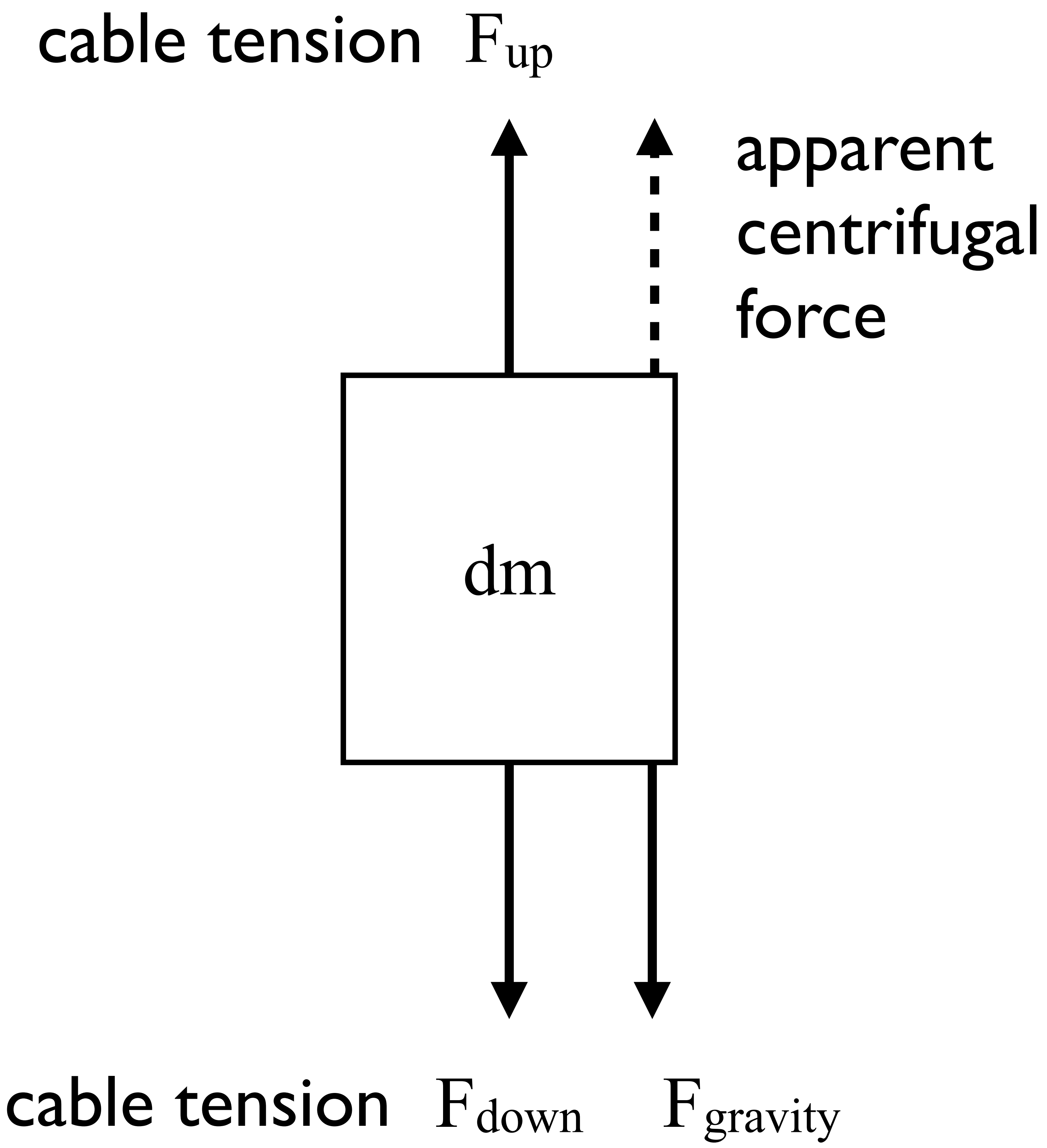
$$r_{\mathrm{GEO}}^3 = \frac{GMP^2}{4\pi^2}$$

	[kg]	[km]	[solar days]	[km]	
	mass	radius	rotper	rgeo	rgeo/r
sun	1.9890e+30	695700.0	25.0462963	25064824.06	36.028208
earth	5.9736e+24	6378.0	0.9972811	42167.82	6.611449
moon	7.3420e+22	1738.1	27.3216667	88436.93	50.881380
mars	6.4185e+23	3396.0	1.0259568	20429.35	6.015711

The cable

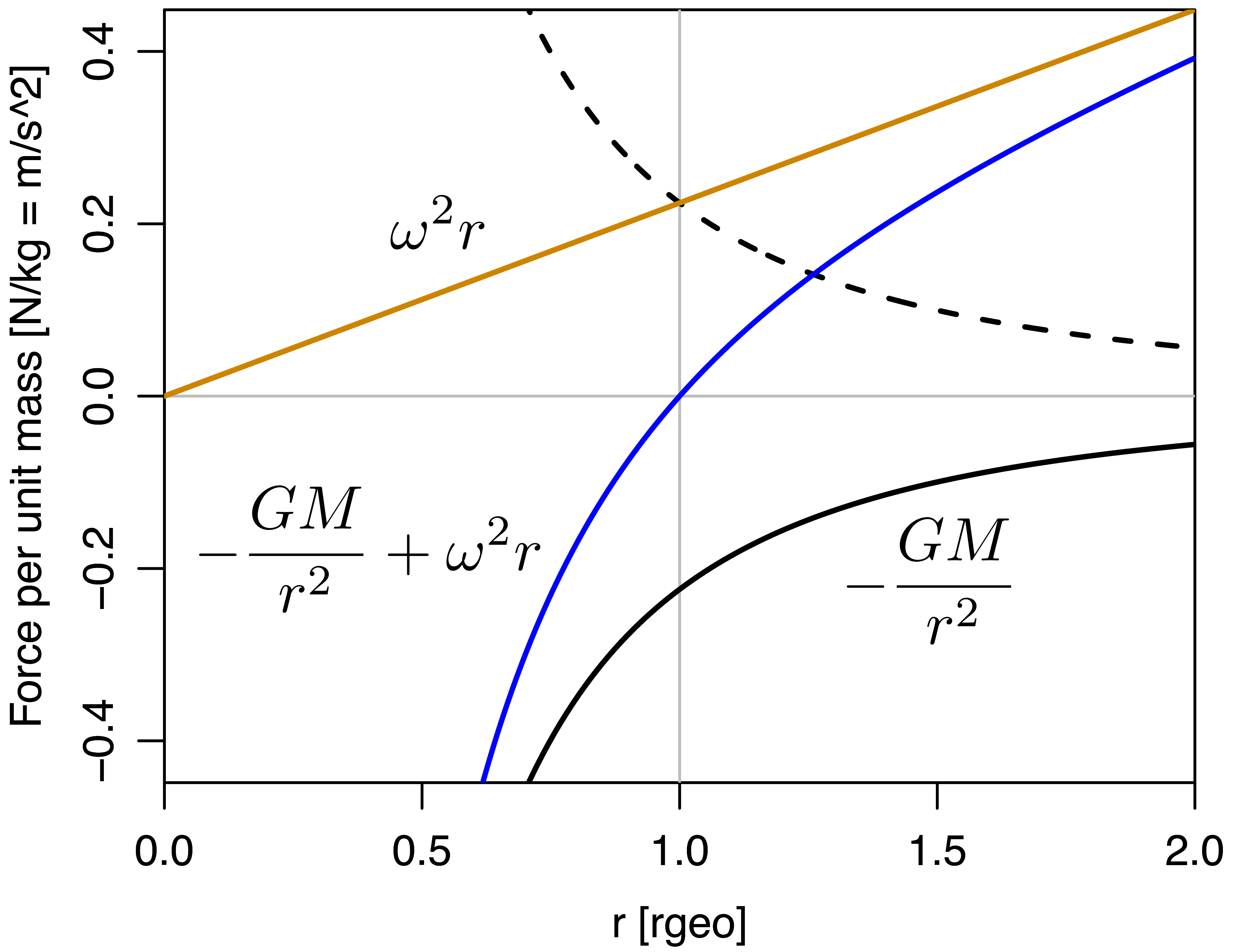


Consider an element of the cable of mass dm

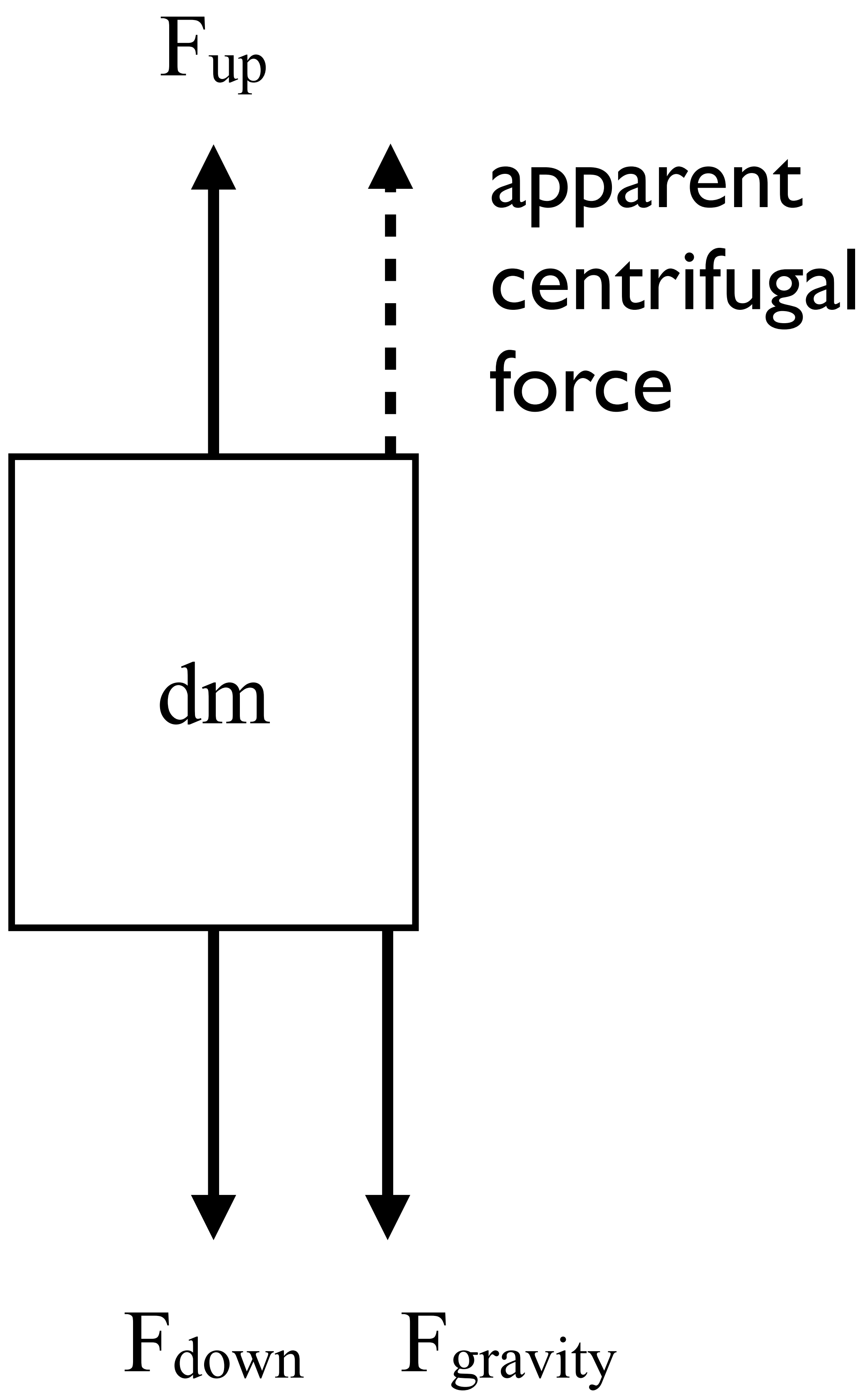


External forces

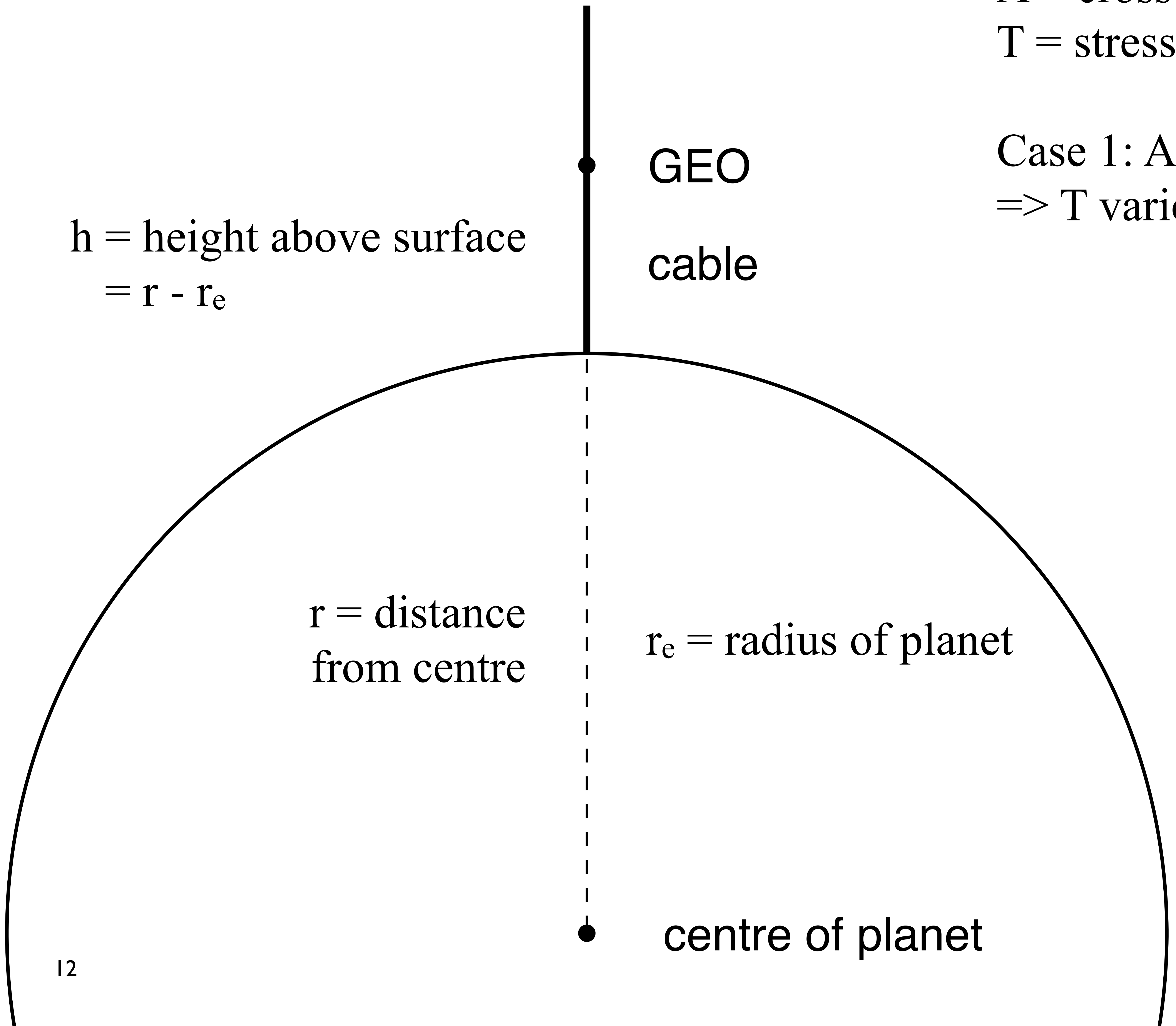
- Above GEO: element is rotating too *fast* for Keplerian orbit
- Below GEO: element is rotating too *slowly* for Keplerian orbit



for $\omega = \dfrac{2\pi}{P_{\rm earth}}$

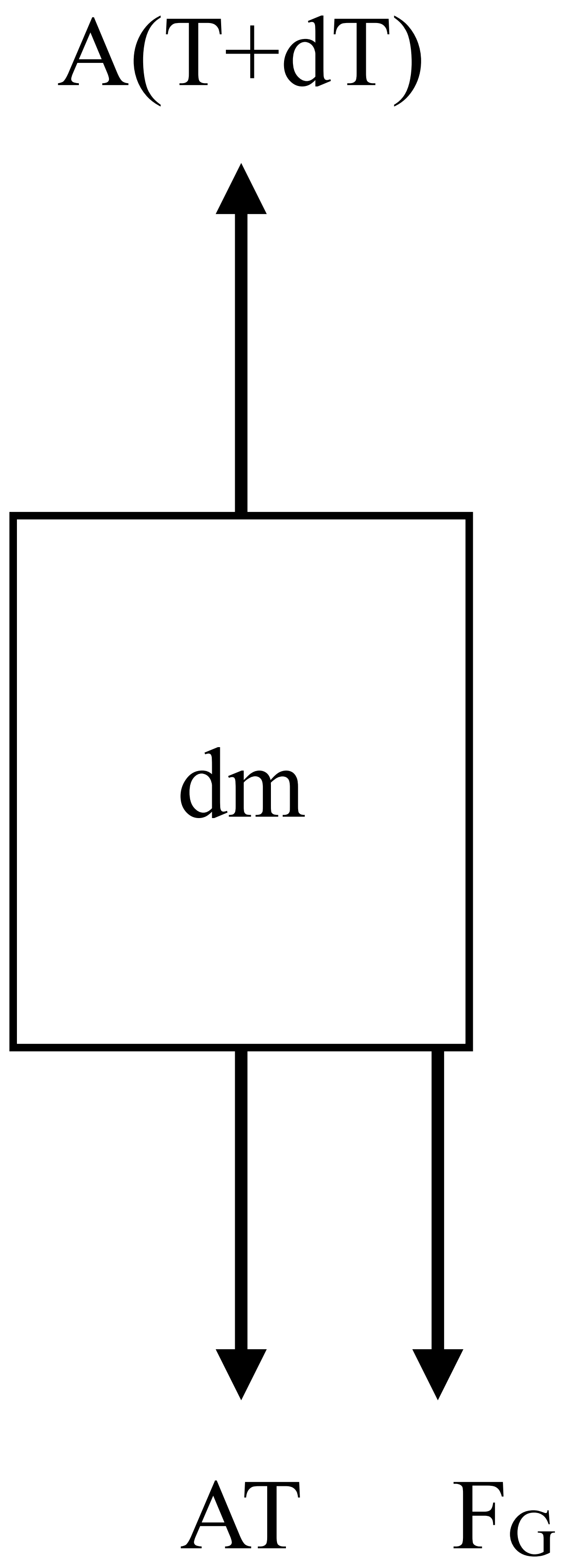


Equation of motion



dm = mass of cable element
A = cross-sectional area of cable
T = stress (force/area) in cable

Case 1: A is constant
=> T varies along cable

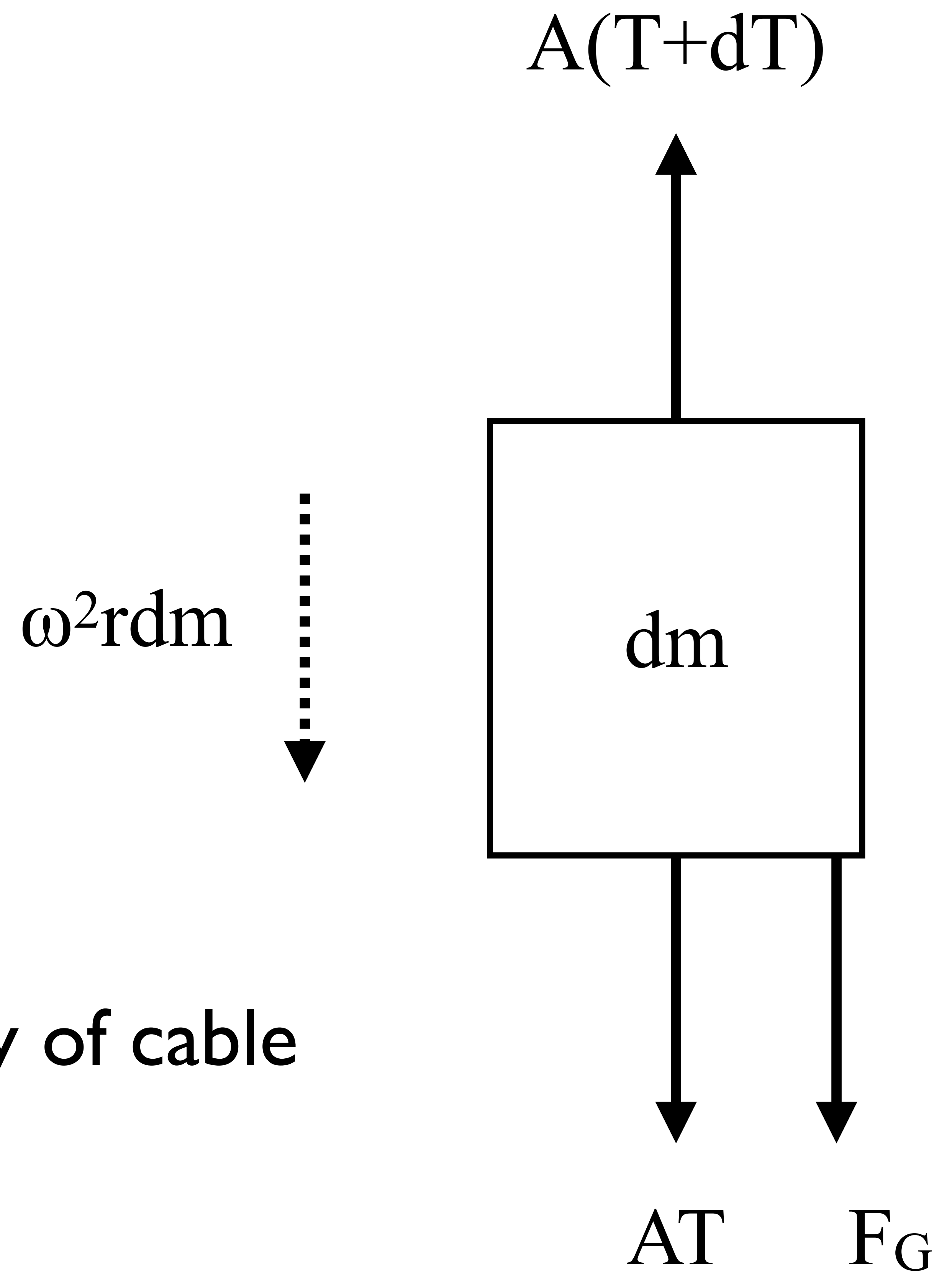


Equation of motion

dm = mass of cable element
A = cross-sectional area of cable
T = stress (force/area) in cable

Net force acting downwards on cable

$$
\begin{aligned}
F &= F_G + AT - A(T + dT) \\
&= F_G - AdT \\
&= \frac{GMdm}{r^2} - AdT
\end{aligned}
$$



Cable rotates as solid body with angular velocity ω so experiences a centripetal acceleration of $\omega^2 r$

$$\omega^2 rdm = \frac{GMdm}{r^2} - AdT$$

$$dm = \rho Adr$$

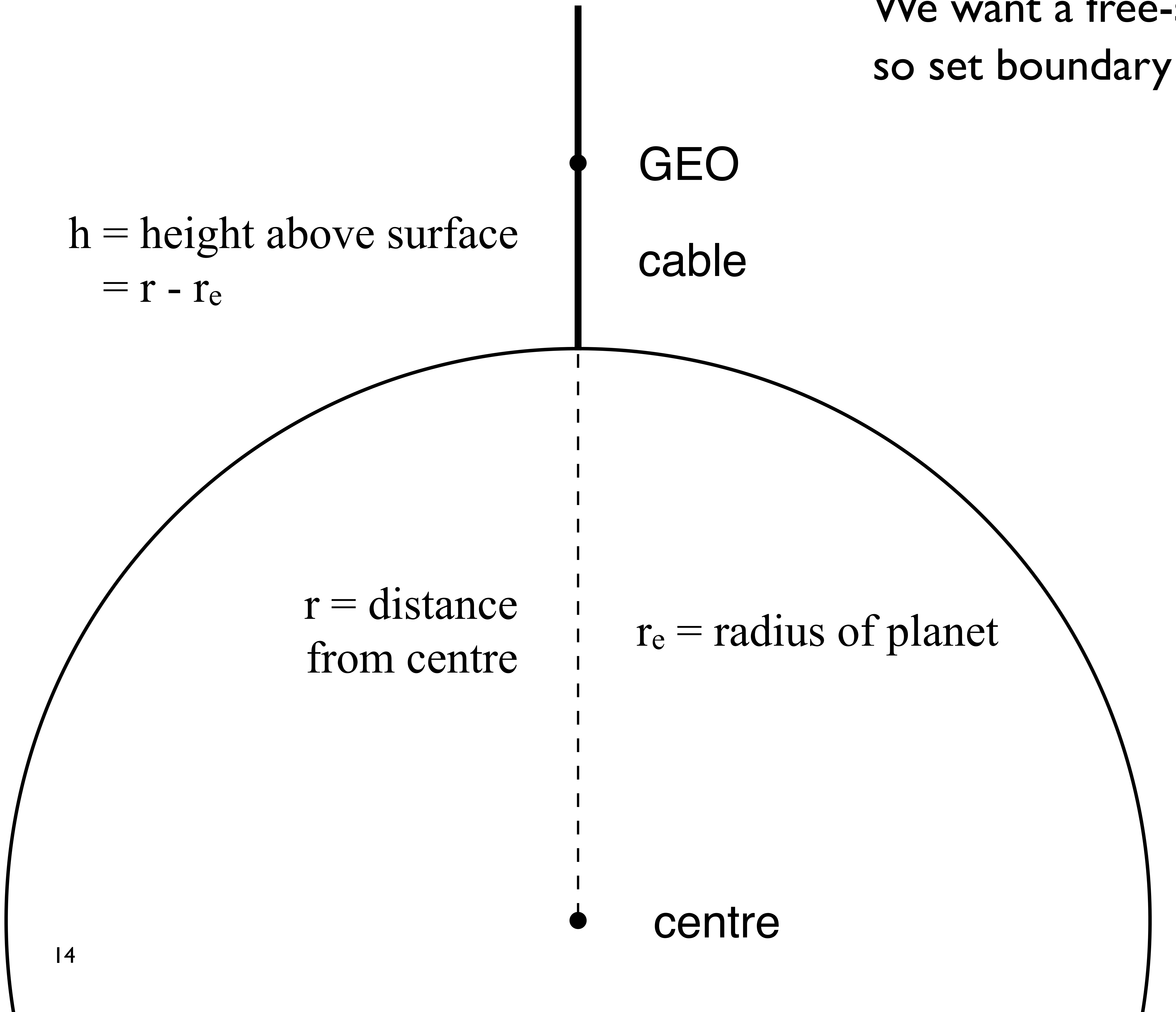
ρ is (constant) density of cable

$$\rho\omega^2 rdr = \frac{GM\rho dr}{r^2} - dT$$

$$\frac{dT}{dr} = \rho\left(\frac{GM}{r^2} - \omega^2 r\right)$$

Equation of motion

Solve by integration.
We want a free-standing cable,
so set boundary condition $T(r=r_e) = 0$



$$\frac{dT}{dr} = \frac{GM\rho}{r^2} - \rho\omega^2 r$$

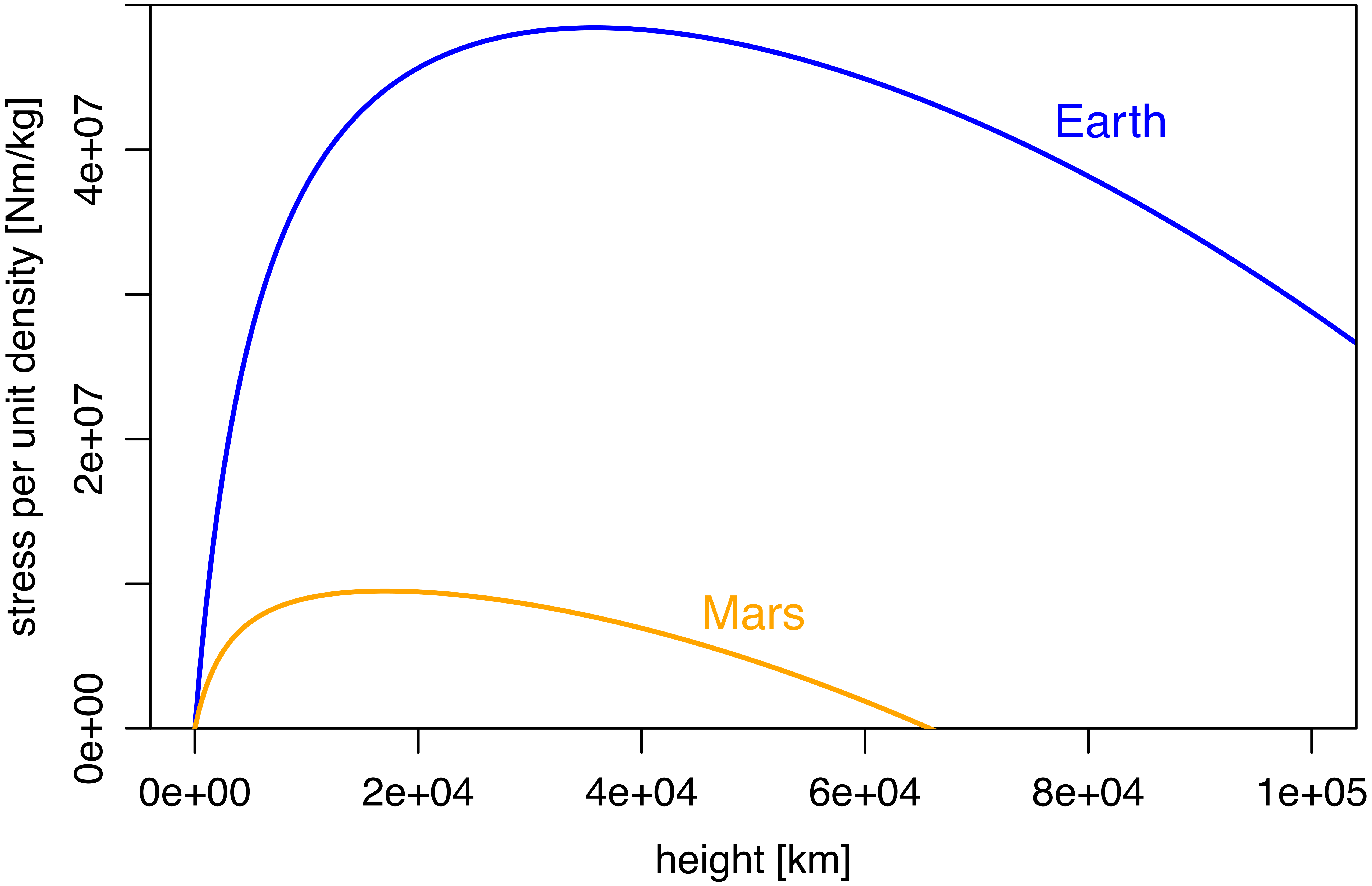
$$\int_0^T dT' = \int_{r_e}^{r} \left(\frac{GM\rho}{r'^2} - \rho\omega^2 r' \right) dr'$$

$$= \ \ldots$$

$$T(h) = \frac{GM\rho h}{r_e(h+r_e)} - \frac{1}{2}\rho\omega^2 h(h+2r_e)$$

Variation of stress per unit cable density with height

$$\frac{T(h)}{\rho} = \frac{GMh}{r_e(h+r_e)} - \frac{1}{2}\omega^2 h(h+2r_e)$$



Where is $\mathrm{T}(\rho)$ maximum?

Find by differentiation:

$$h + r_e = \left(\frac{GM}{\omega^2}\right)^{1/3} = r_{\mathrm{GEO}}$$

How long does the cable have to be?

- For a free-standing cable, set boundary condition $T(h=L) = 0$

$$0 = \frac{GML}{r_e(L+r_e)} - \frac{1}{2}\omega^2 L(L+2r_e)$$

cubic equation, but becomes quadratic when we ignore the $L=0$ solution

$$= \ldots$$

$$L = -\frac{3r_e}{2} \pm \sqrt{\frac{r_e^2}{4} + \frac{2GM}{\omega^2 r_e}}$$

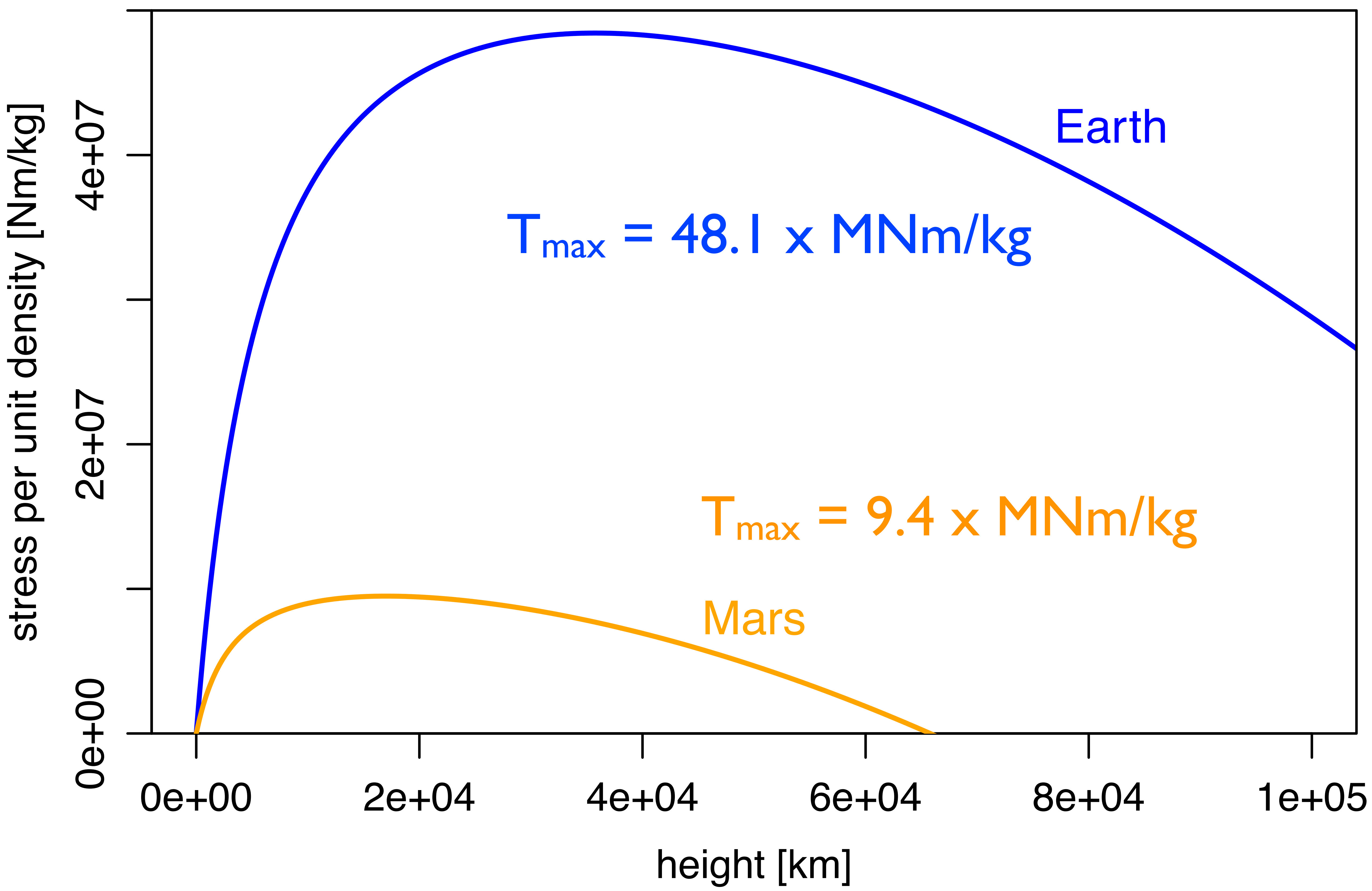
$$L = \frac{r_e}{2}\left(\sqrt{1+8\left(\frac{r_{\mathrm{GEO}}}{r_e}\right)^3} - 3\right)$$

= 144 000 km for the Earth (= 3.41 r_{GEO})

How large is the stress?

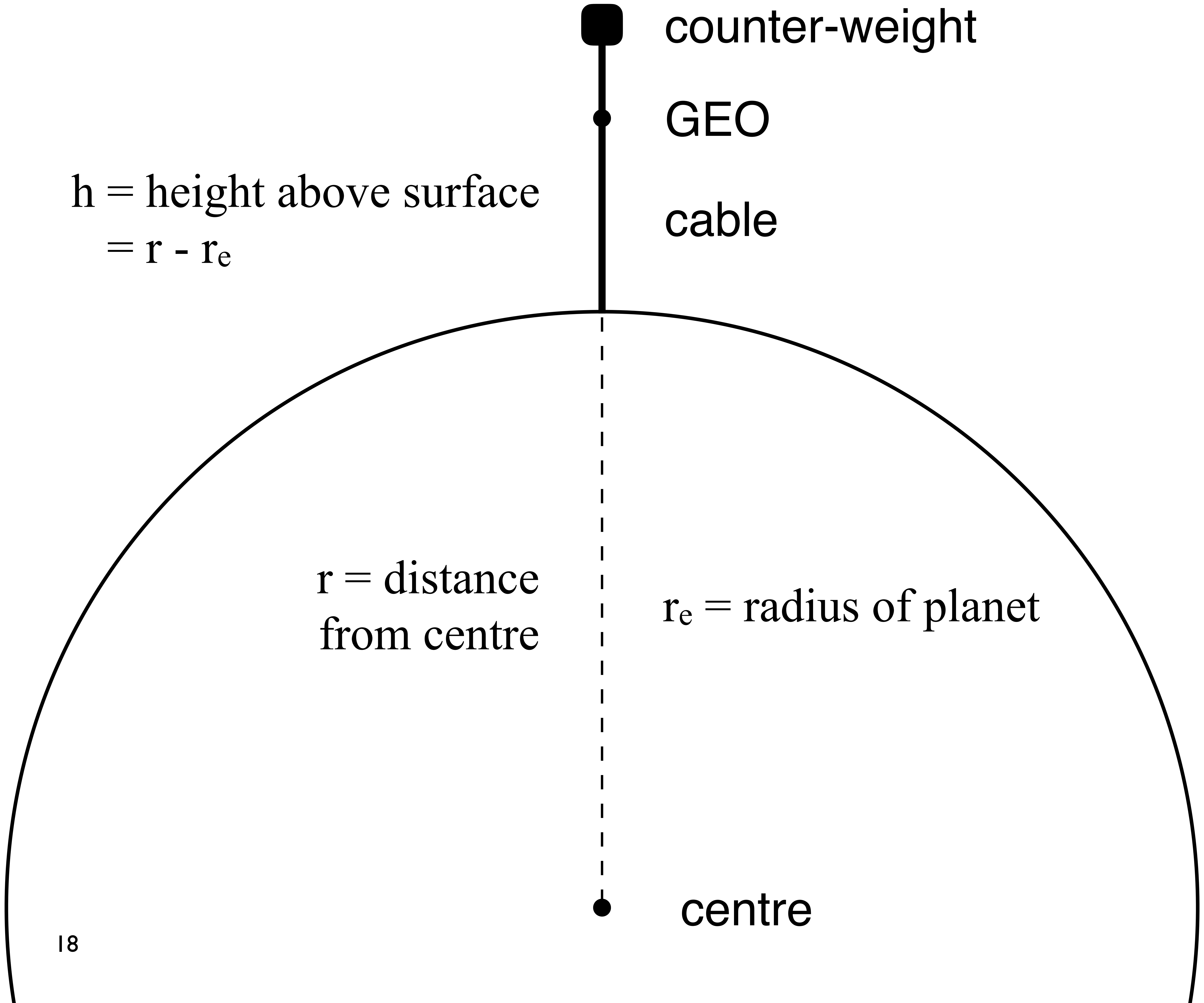
$$\frac{T(h)}{\rho} = \frac{GMh}{r_e(h + r_e)} - \frac{1}{2}\omega^2 h(h + 2r_e)$$

specific strength = maximum stress per unit density



	density	strength	specific strength
	kg m^{-3}	10^9 Pa	MNm kg^{-1}
steel	8000	2.5	0.3
silicon	3300	7	2.1
kevlar	1440	3.6	2.5
spider silk	1300	1.0	0.8
carbon nanotube	1300	130	100

Counter-weight

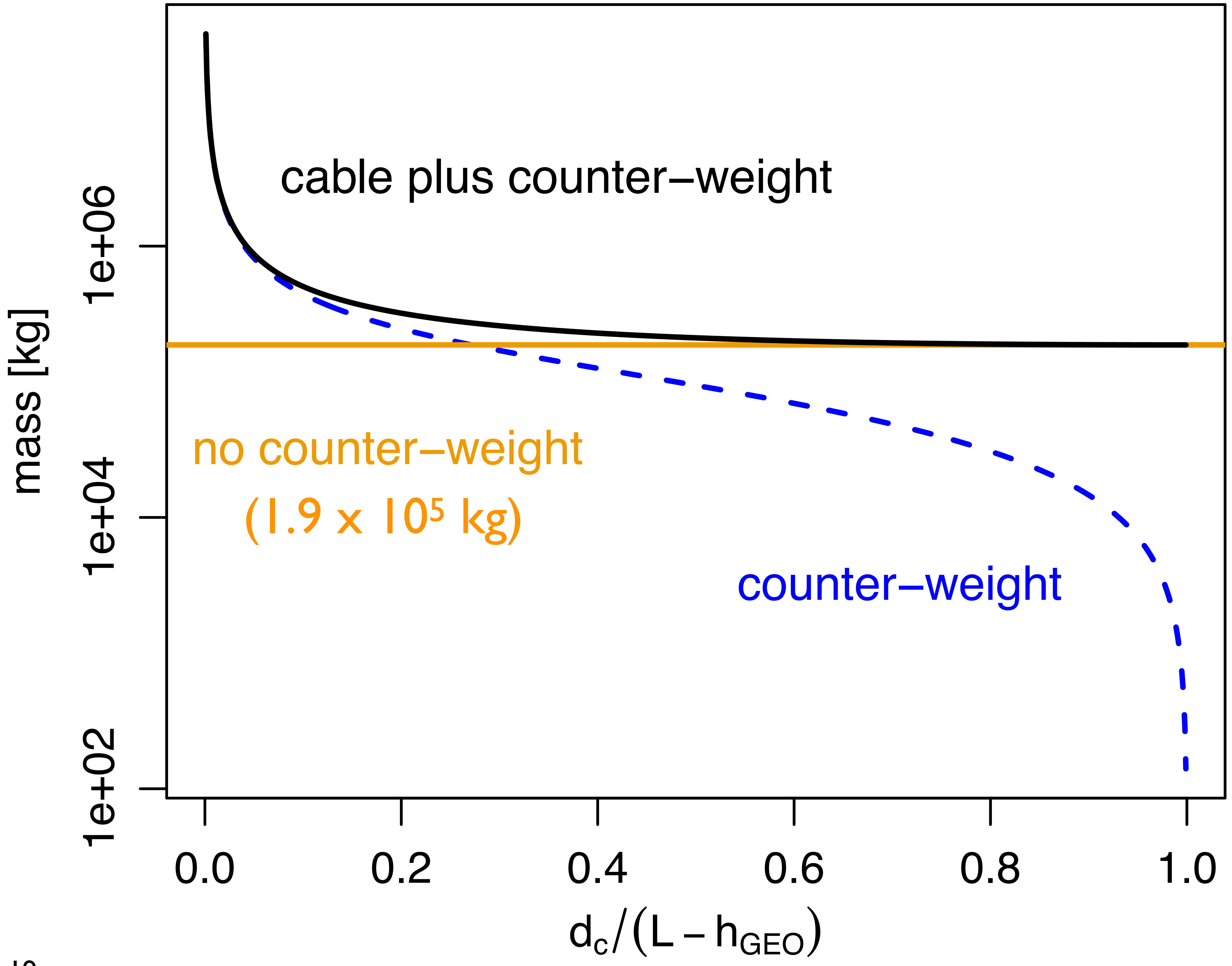


- Can use a shorter cable by using a counter-weight
- This must be at $r=r_c > r_{GEO}$
- Compute mass m_c by balancing forces:

$$\frac{GMm_c}{r_c^2} + AT(h=r_c-r_e) = m_c\omega^2 r_c$$

Counter-weight

For the Earth with carbon nanotube cable with $A = 10^{-5}$ m^2 (e.g 10mm x 1mm)



- Can use a shorter cable by using a counter-weight
- This must be at $r = r_c > r_{GEO}$
- Compute mass m_c by balancing forces:

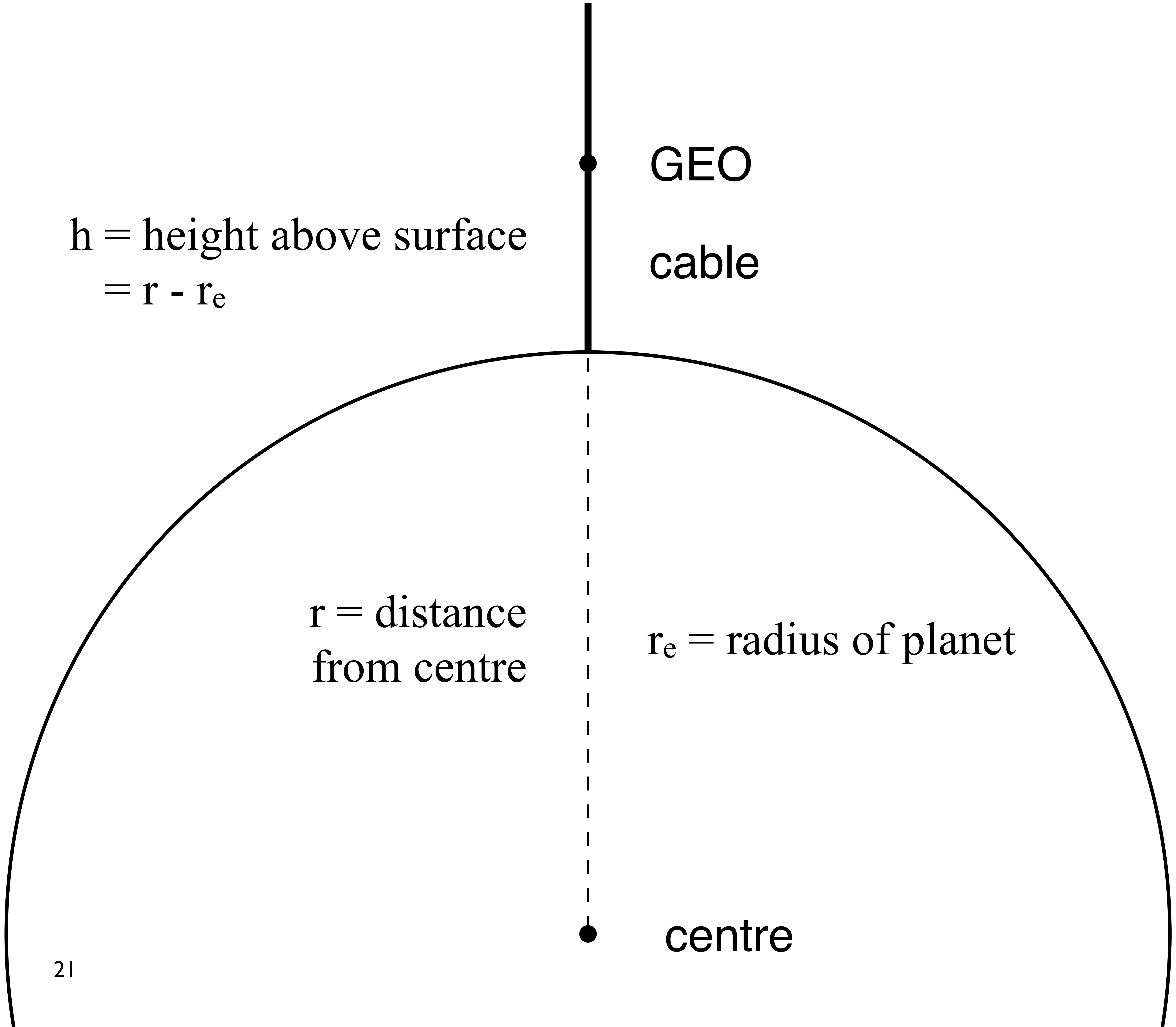
$$\frac{GMm_c}{r_c^2} + AT(h = r_c - r_e) = m_c\omega^2 r_c$$

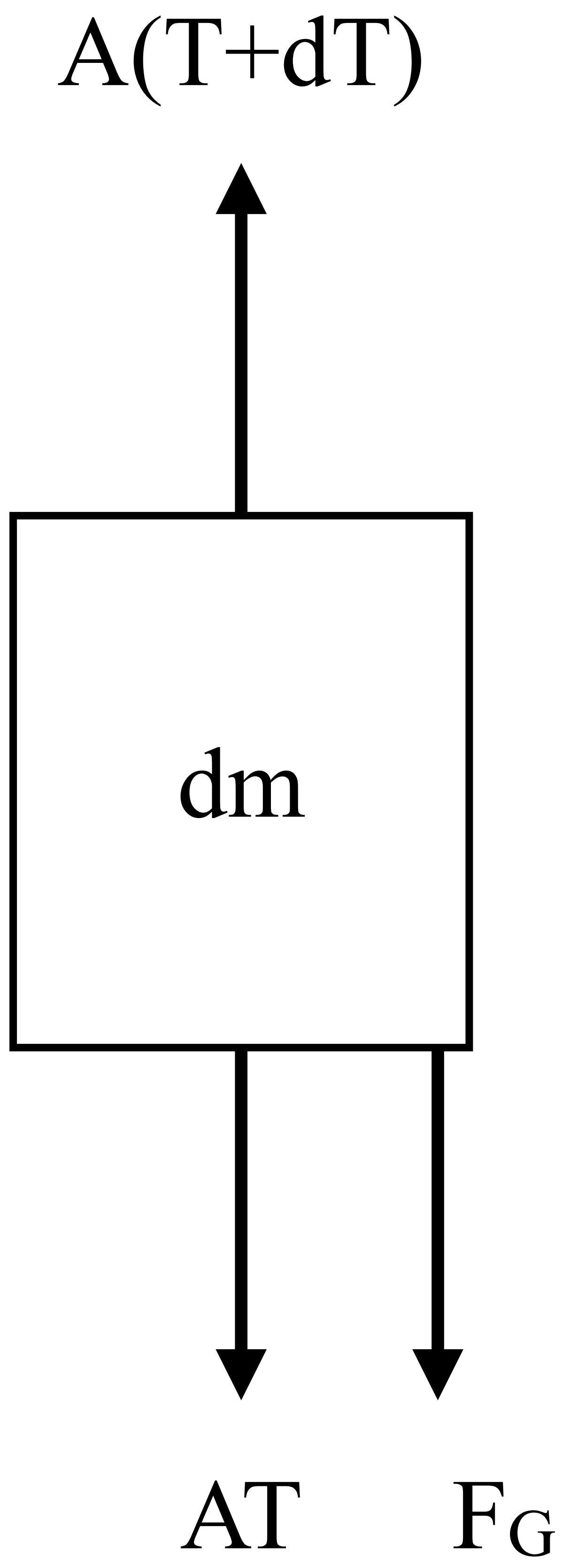
$d_c = r_c - r_{GEO}$ = height of counter-weight above GEO
L = length of cable without counter-weight

Important points

- Cable is in equilibrium between gravity and centrifugal force
  - Pull in opposite directions, so cable is under tension
- Cable has to “hold up” its own weight, so it needs to be strong and light
  - high specific strength (= yield strength per unit density)
- Most materials are too weak, so cannot be used, not even in principle
  - carbon nanotubes work in principle, but don't yet exist in bulk
- A counter-weight means you need less cable material, but not less mass
  - must be placed at correct altitude for its mass

So far: constant cross-section area





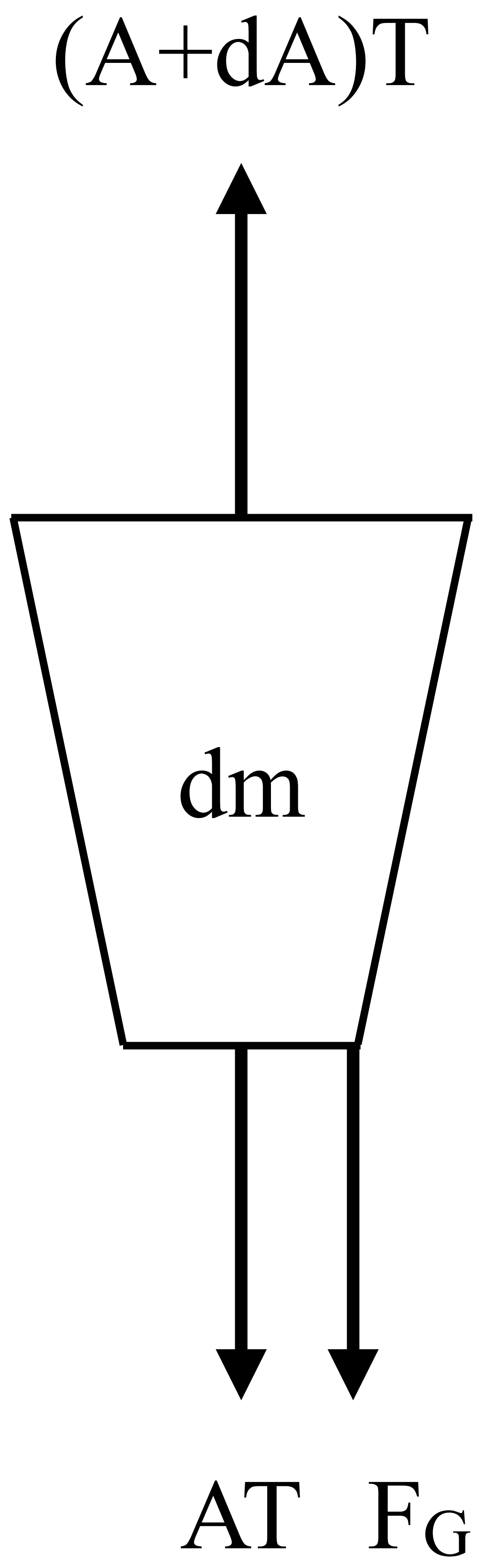
Tapered cable

Now keep T fixed and allow A to vary

$$dm = \rho \left(\frac{(A + dA) + A}{2} \right) dr \simeq \rho A dr$$

$$\frac{dA}{dr} = \frac{A\rho}{T} \left(\frac{GM}{r^2} - \omega^2 r \right)$$

cf. A fixed, T varies: $\frac{dT}{dr} = \rho \left(\frac{GM}{r^2} - \omega^2 r \right)$

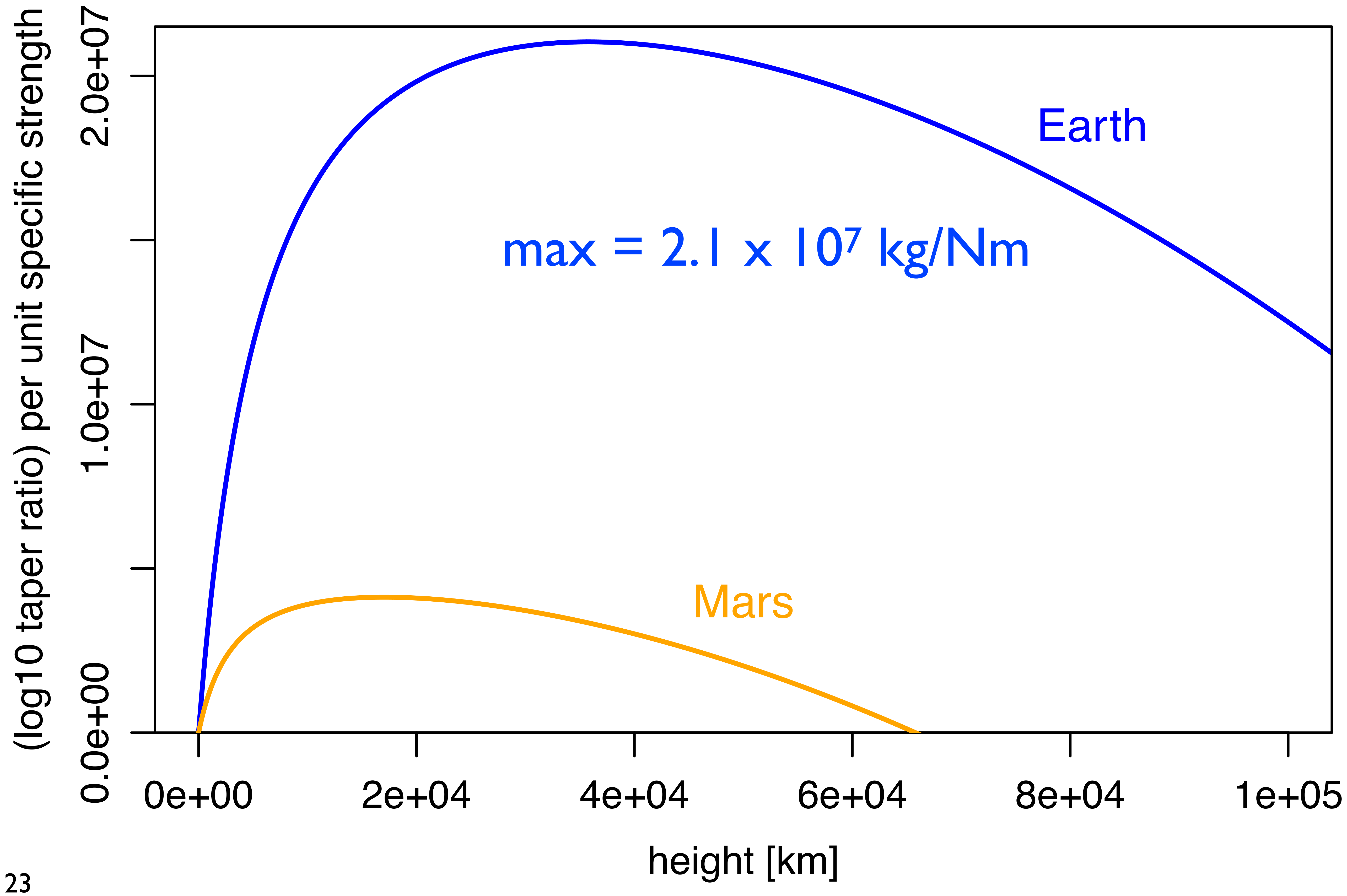


Solve by integration with boundary condition $A(r=r_e) = A_e$

$$\ln \frac{A}{A_e} = \frac{\rho}{T} \left[GM \left(\frac{1}{r_e} - \frac{1}{r} \right) - \frac{1}{2}\omega^2 (r^2 - r_e^2) \right] \quad \text{where } r \geq r_e$$

Tapered cable

$$\ln \frac{A}{A_e} = \frac{\rho}{T}\left[GM\left(\frac{1}{r_e} - \frac{1}{r}\right) - \frac{1}{2}\omega^2(r^2 - r_e^2)\right] \quad \text{where } r = h + r_e$$



The taper ratio depends exponentially on the specific strength

Steel: $\frac{T}{\rho} = 3 \times 10^5 \;\Rightarrow\; \frac{A_{\mathrm{GEO}}}{A_e} \sim 10^{33}$

Kevlar: $\frac{T}{\rho} = 2.5 \times 10^6 \;\Rightarrow\; \frac{A_{\mathrm{GEO}}}{A_e} \sim 10^{8}$

Carbon nanotubes:

$\frac{T}{\rho} = 1 \times 10^8 \;\Rightarrow\; \frac{A_{\mathrm{GEO}}}{A_e} \sim 10^{0.2} \simeq 1.6$

Tapered cable

- A tapered cable is lighter than one of constant cross-sectional area => cheaper
- Has the same length as the untapered one, but can also be used with a counter-weight
- Maximum taper ratio (GEO to ground) for carbon nanotubes will be larger than 1.6 in practice, due to imperfections, design margins etc.
- Aravind (2006) adopts a maximum taper ratio of 4.3
  - $A_e = 1.5 \times 10^{-7}$ m^2
  - this gives a mass of 100×10^3 kg for cable and 53×10^3 kg for a counter-weight at 10^5 km above ground, i.e. with $d_c = 0.59(L - h_{GEO})$
  - can support climbers of up to 10^3 kg

Summary of part 1

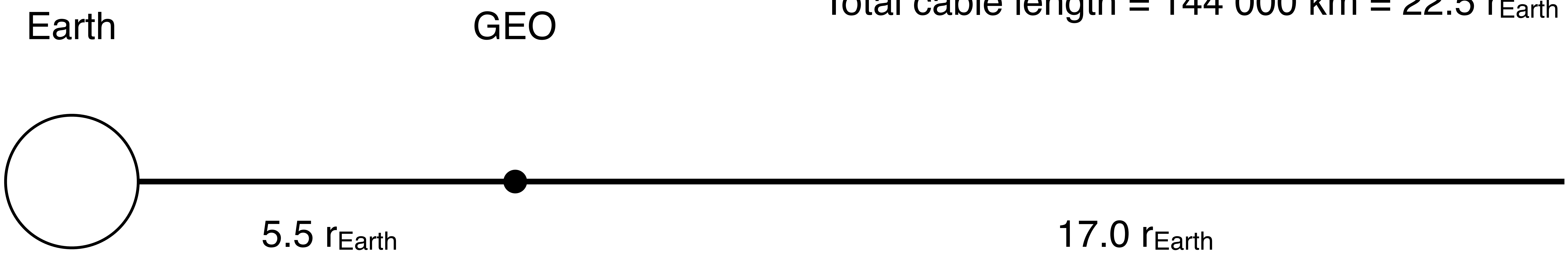
- High tension cable extending to beyond GEO
  - balance between gravitational and centrifugal forces
- For equilibrium (i.e. remain standing up)
  - cable of specific length, or a counter-weight of specific mass, position
- Requirement: materials of sufficiently high specific tensile strength (carbon nanotubes)
- Use a tapered cable to minimize cable mass
  - i.e. keep stress near maximum along its entire length by varying cross-section
  - then don't need large masses of material: ~ 100 tonnes cable plus counter-weight

Part 2

The space elevator is big: to scale (no counter-weight)

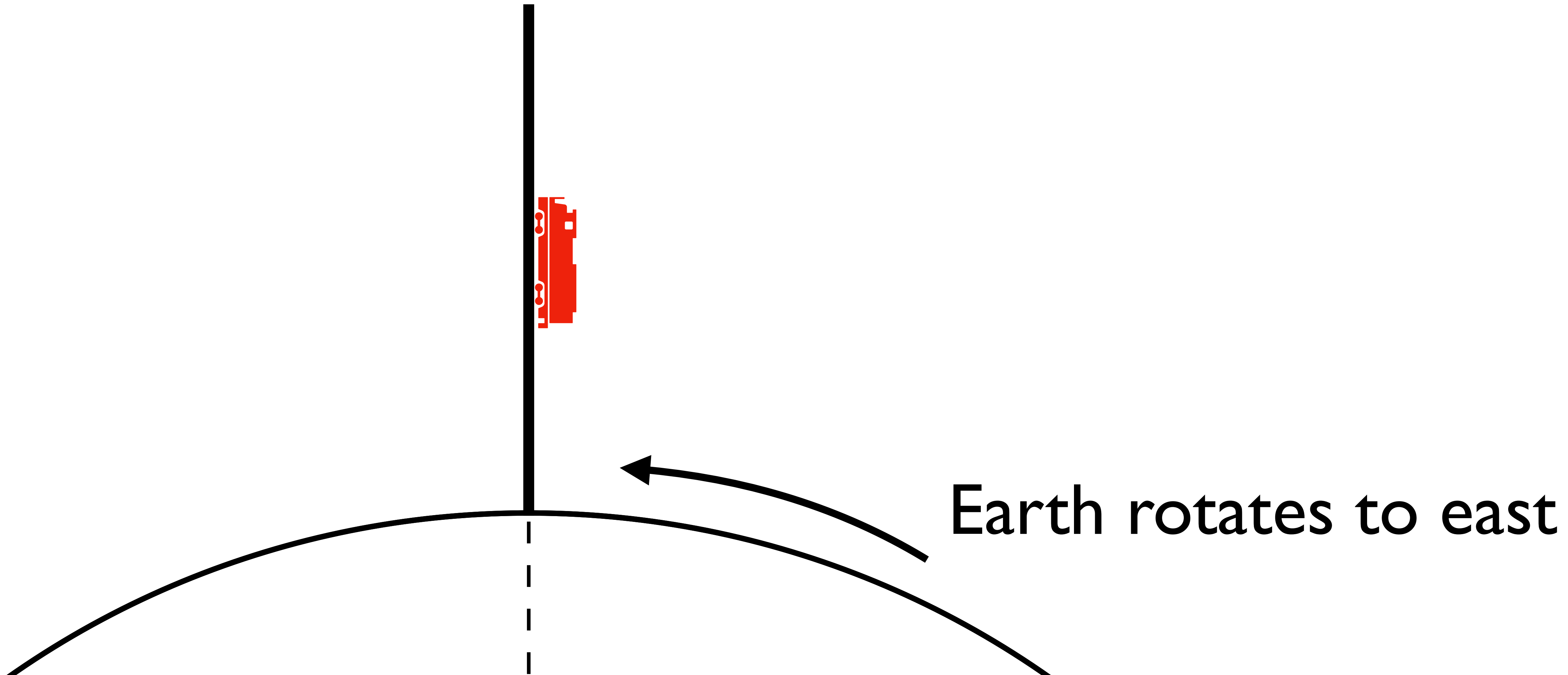
Earth radius = 6400 km

Total cable length = 144 000 km = 22.5 r_{Earth}



Climbers

- Vertical railway (obviously don't use rockets or cables!)
- At 100 m/s (360 km/h), it would take 4.1 days to reach GEO
- 10^4 kg climber exerts (gravitational) force of 10^5 N on cable at surface
  - carbon nanotube needs to have $A > 10^{-6}$ m^2 to support this (e.g. 10mm x 0.1 mm)
- Ascending climber deflects cable to west due to Coriolis force $2\mathbf{v} \times \mathbf{\Omega}$
  - climber moves onto cable portion that has larger horizontal velocity, so gets "left behind"



Climbers: Energy required

- Only need to supply energy to move climber up
  - i.e. to increase its potential energy
- In contrast to a rocket, we don't also need to provide required orbital speed
  - this, and the angular momentum, is provided by the cable, i.e. the Earth
- How much of an energy saving is this?

Specific energy (i.e. per kg) to GEO

Potential energy of body on planet's equator: $$U_{\rm surface} = -\frac{GM}{r_e}$$ -62.5 MJ

Total energy of body at rest on planet's equator: (not a Keplerian orbit!) $$E_{\rm surface} = -\frac{GM}{r_e} + \frac{1}{2}\underbrace{\left(\frac{2\pi r_e}{P}\right)^2}_{0.1\ \rm MJ}$$ -62.4 MJ

Potential energy of body in GEO: $$U_{\rm GEO} = -\frac{GM}{r_{\rm GEO}}$$ -9.5 MJ

Total energy of body in GEO orbit (Keplerian): $$E_{\rm GEO} = -\frac{GM}{2r_{\rm GEO}}$$ -4.7 MJ

$$\Delta E_{\rm ideal\ rocket} = E_{\rm GEO} - E_{\rm surface}$$ 57.7 MJ

$$\Delta E_{\rm elevator} = U_{\rm GEO} - U_{\rm surface}$$ 53.1 MJ

values for the Earth

Climbers: Energy required

- The elevator still has to supply 92% of the energy of an "ideal" rocket
  - an "ideal" rocket is one that accelerates instantly from Earth's surface
  - but in practice, most of the energy of a rocket goes into lifting the fuel, tanks, engines etc.
    - payload of a rocket launched to GEO is just a few percent of the total mass:

$$\frac{M_i}{M_f} = \exp\left(\frac{\Delta V}{v_e}\right) = \exp\left(\frac{14}{4.5}\right) = 22$$

  - so we actually need far less energy for the climbers per unit mass payload
- Additional energy needed in practice also for the climbers
  - air resistance: ~ v^2, so is greater for rocket
  - energy conversion losses (for both); friction on cable (for elevator)

Climbers: Energy supply

- 10^4 kg climber moving at 100 m/s needs 10 MW power at Earth's surface ($P = mgv$)
- Onboard power plant would be too heavy
- Provide energy remotely
  - carbon nanotubes are conductors
  - use (powerful) lasers with conversion to electricity on the climber
    - to achieve focus, need a large array or adaptive optics
    - we will look at lasers in the lecture on laser sails

Launching spacecraft from the cable

- This is the main purpose: to launch spacecraft without having to lift the fuel (or the rocket itself) to get it into orbit
  - recall that most of the fuel used by a rocket is to lift the remaining fuel
  - can therefore launch order of magnitude more payload for the same energy
- We could lift rocket parts to a space station at GEO and launch from there
- But why stop at GEO if the cable extends further?

Launching spacecraft at rest from the cable

Minimum velocity required for launch:
(i.e. to leave Earth's gravitational field)

$$v_{\rm escape} = \sqrt{2} v_{\rm circular} = \sqrt{\frac{2GM}{r}}$$

Actual velocity of elevator at radius r:

$$v = \omega r$$

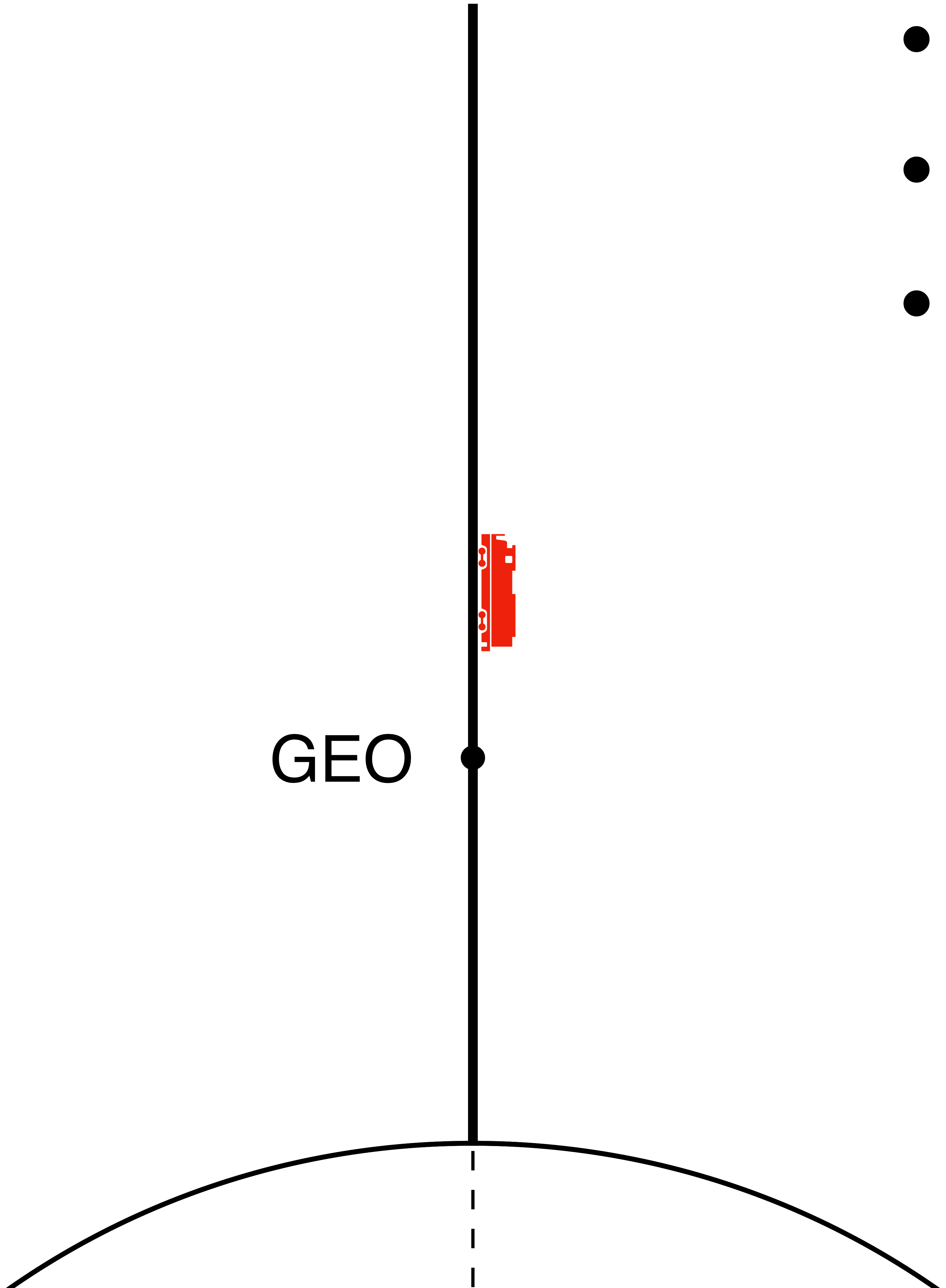
Equate these and solve for r:

$$v = v_{\rm escape}$$

$$r^3 = \frac{2GM}{\omega^2}$$

$$r = 2^{1/3} r_{\rm GEO}$$ = 1.26$r_{\rm GEO}$ = 53 130 km for Earth
(10 960 km above GEO)

Launching spacecraft at rest from the cable

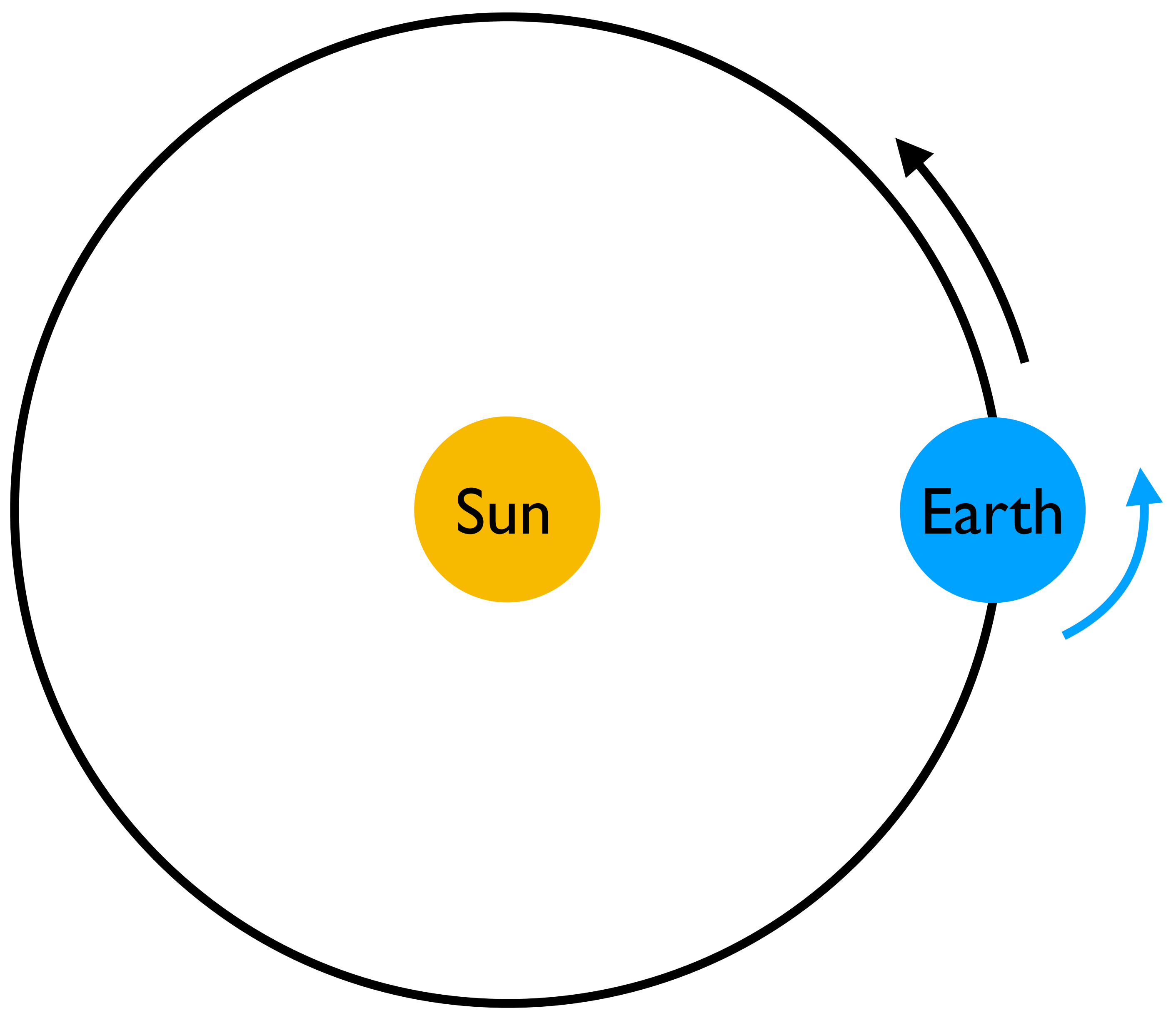


- Consider releasing from rest relative to cable
- At $r=1.26r_{GEO}$, $v=3.87$ km/s relative to Earth's centre
- At full length of cable (no counter-weight)
  - $r=3.56r_{GEO}$ (150 000 km)
  - $v=10.95$ km/s relative to Earth's centre*** (cf. $v_{escape} = 2.3$ km/s)

***close to - but not equal to - the escape velocity from Earth's surface

Launching spacecraft from the cable

- Escape velocity of Sun at Earth = $\sqrt{2}\, v_{circular} = 42.1$ km/s
- If launch at 00:00 local solar time, gain full orbital speed of Earth: $11.0+29.7 = 40.7$ km/s
    - orbit around Sun has perihelion at Earth, and aphelion further away
    - enough energy to reach outer solar
    - control direction by day of year of release
- Control orbit also by time of day of release
- If launch during day, perihelion will be within Earth's orbit
    - i.e. can fly payloads to the inner planets



Better: climber to GEO and then let it slide up cable…

- Can write down equation-of-motion that includes Coriolis and centrifugal force
- Solution for r(t) and v(t) involves nasty integrals
- But solution for v(r) is easier

$2\mathbf{v} \times \mathbf{\Omega}$ is perpendicular to cable so we can ignore it

Re-write acceleration: $$\ddot{r} = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$$

Equation of motion: $$v\frac{dv}{dr} = \omega^2 r - \frac{GM}{r^2}$$

Initial condition: $$v(r = r_0) = v_0$$

Solve by integration: $$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right) + \omega^2(r^2 - r_0^2)$$

Release climber at velocity v_0 at some distance r_0 (> r_{GEO}).

Cable rotation accelerates climber up the cable.

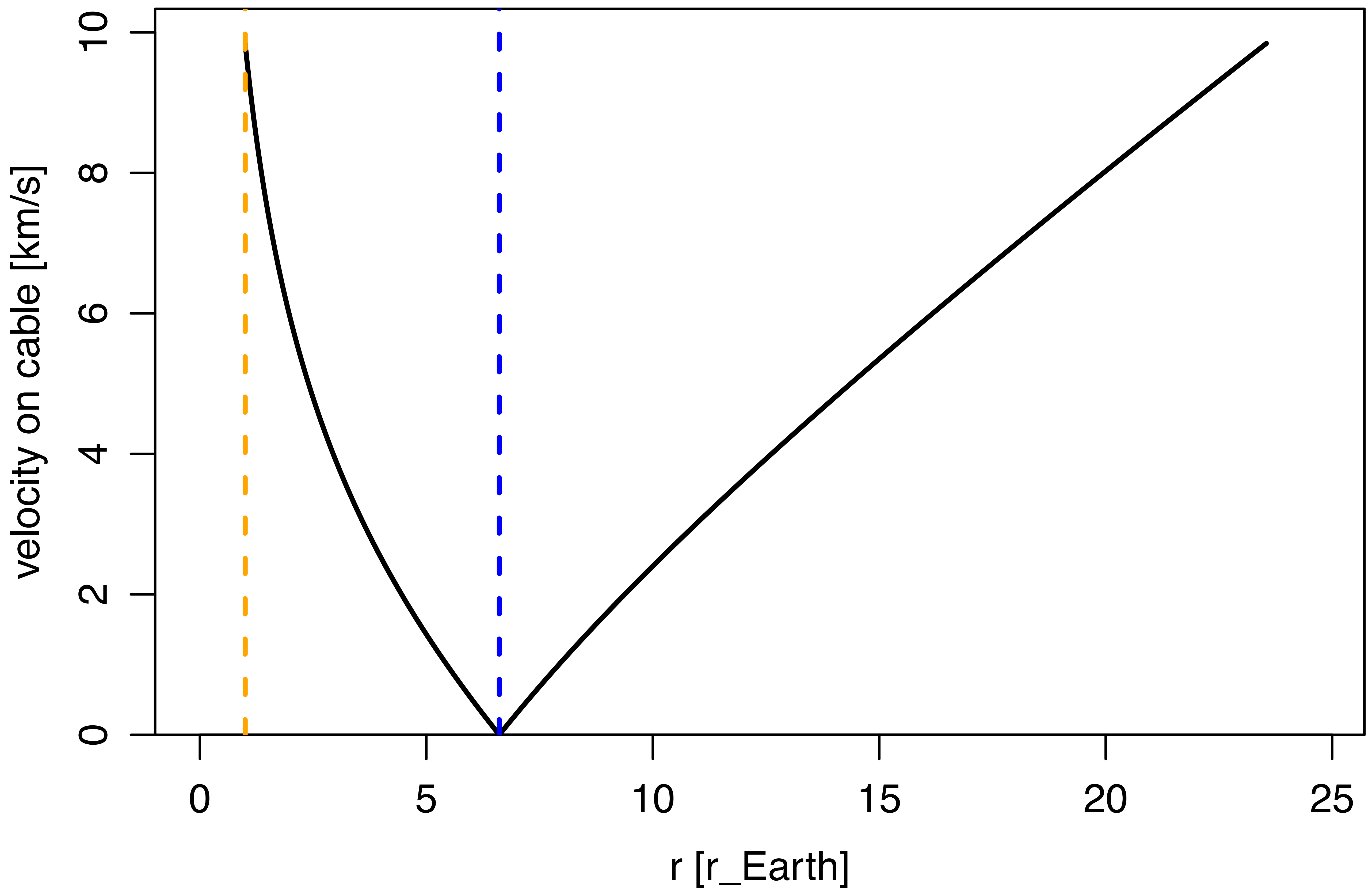
Calculate velocity v at some point r higher up the cable.

Launching spacecraft using rotation of cable

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right) + \omega^2(r^2 - r_0^2)$$

$r_0 = r_{\mathrm{GEO}}$ blue dotted line

$v_0 = 0$



Release from end of longest cable (144 000 km). Speeds relative to Earth's centre:

v radial = 9.84 km/s *(what we computed)*

v tangential = 10.95 km/s *(of cable)*

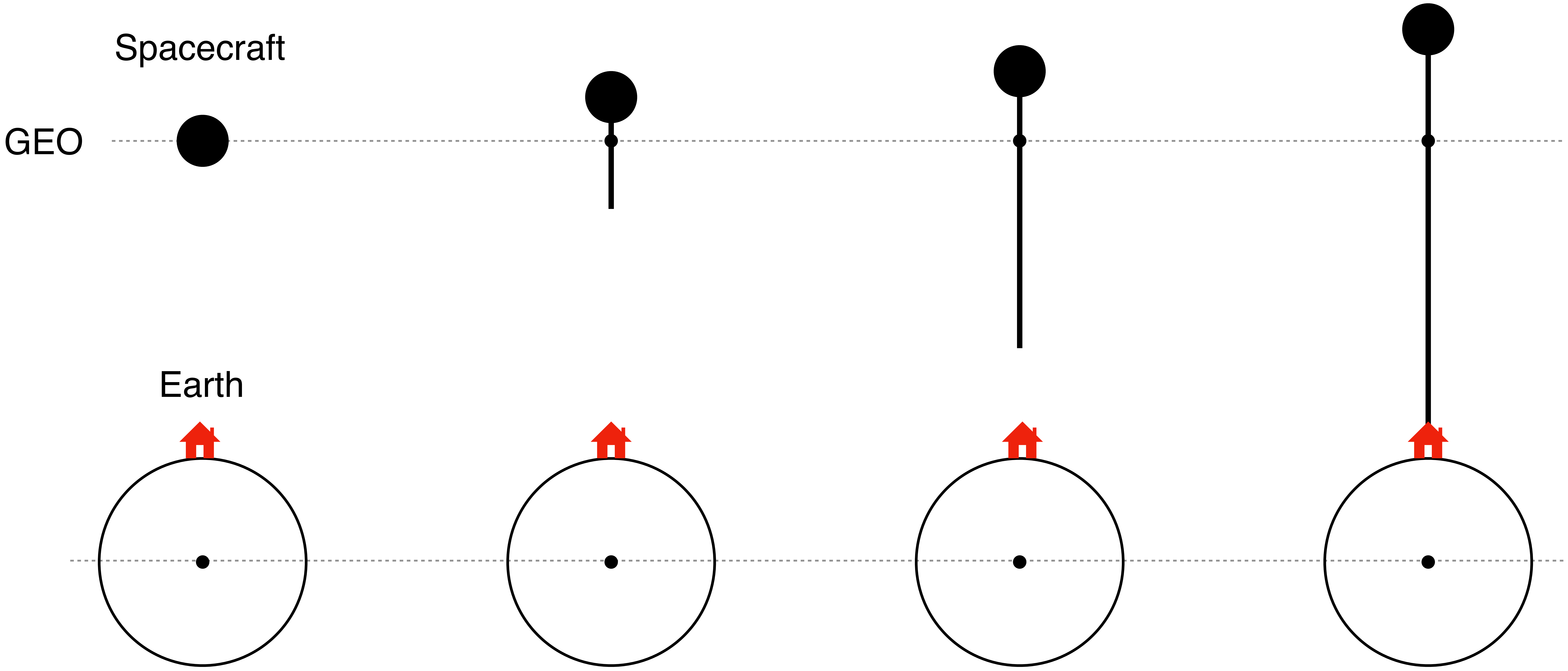
v total = 14.72 km/s *(quadrature sum)*

Launch angle to cable = 48°

How to build the elevator

- Cannot build from the ground up, as the cable cannot support compression
- Deploy wound-up cable from a satellite at GEO
- Two choices. In both cases centre-of-mass does not move
  - satellite at GEO deploys simultaneously up and down at different rates, to keep satellite at GEO
  - satellite deploys cable down: satellite moves up simultaneously (it can become the counter-weight)
- Only have to actively deploy initially: gravity and centrifugal forces will continue it

Cable deployment



Ground connection

- Must be on equator
  - land or ocean
  - avoid stormy regions
- Tension from cable alone is nearly zero
  - but is larger due to climbers

Challenges and risks

- Weather (wind, lightning)
- Erosion by atomic oxygen at high altitudes
- Collision with aircraft, satellites, space debris, meteors
  - can track larger objects by radar
  - need to move the cable (GEO station) now and then to avoid large objects (so put on the ocean or a railway track)
- Damage by micrometeorites
  - cable as a wide, thin tape, with narrow profile in direction of Earth rotation
  - cable would need repairing, and eventually replacing
- What happens if the cable breaks?

Summary of part 2

- Space elevator overcomes limitations of rockets for launches
  - payload ratios *much* lower => less energy => lower cost
  - better reusability, faster launch rates, can bring things down from orbit too
  - for the launch, steal energy and angular momentum from the Earth's rotation
- With the full length cable, can give payloads a velocity relative to the Earth's centre of
  - Δv = 11.0 km/s if launching from rest (i.e. just from rotation speed of the cable)
  - Δv = 14.7 km/s if letting spacecraft slide up from GEO (initially at rest at GEO)
- Can control direction of launch according to time of day/year, and length of cable